

Kummer Subspaces of Tensor Products of Cyclic Algebras

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Abstract

We discuss the Kummer subspaces of tensor products of cyclic algebras, focusing mainly on the case of cyclic algebras of degree 3. We present a family of maximal spaces in the general case, classify all the monomial spaces in the case of tensor products of cyclic algebras of degree 3 using graph theory, and provide an upper bound for the dimension in the generic tensor product of cyclic algebras of degree 3.

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1. Introduction

Let p be a prime number and F be an infinite field of characteristic not p containing a primitive p th root of unity ρ . A cyclic algebra of degree p over F is an algebra that can be presented as

$$F[x, y : x^p = \alpha, y^p = \beta, yx = \rho xy]$$

for some $\alpha, \beta \in F^\times$. We denote the presentation as $(\alpha, \beta)_{p, F}$. A given algebra can have more than one presentation. Fixing a presentation, we call the elements $x^i y^j$ where i and j are integers between 0 and $p - 1$ “monomials”. The same goes for tensor products of cyclic algebras, i.e. if we fix presentations $F[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_k x_k = \rho x_k y_k] = (\alpha_k, \beta_k)_{p, F}$ then the monomials in the tensor product $\bigotimes_{k=1}^n (\alpha_k, \beta_k)_{p, F}$ are $\prod_{k=1}^n x_k^{i_k} y_k^{j_k}$.

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Let A be a division tensor product of n cyclic algebras of degree p . An element $x \in A$ is called Kummer (or p -central) if $x^p \in F$ whereas $x^k \notin F$ for every $1 \leq k \leq p-1$. An F -vector subspace of A is called Kummer if all its nonzero elements are Kummer. A necessary and sufficient condition for a space $Fx_1 + \cdots + Fx_m$ to be Kummer is that $x_1^{d_1} * \cdots * x_m^{d_m} \in F$ for every m -tuple of nonnegative integers d_1, \dots, d_m satisfying $d_1 + \cdots + d_m = p$. The expression $x_1^{d_1} * \cdots * x_m^{d_m}$ stands for the sum of all the words in which each x_k appears d_k times, e.g. $x_1^2 * x_2 = x_1^2 x_2 + x_1 x_2 x_1 + x_2 x_1^2$. This notation was introduced in [Rev77]. Fixing the presentations of the cyclic algebras, a Kummer space is called “monomial” if it is spanned by monomials.

The classification of Kummer spaces is an open problem. There has not even been found yet an upper bound for the dimension of such spaces, except for a few special cases. If $p = 2$ and $n = 1$ the space of elements of trace zero contains all the Kummer elements. In general it is known that for $p = 2$ the size of the Kummer space is bounded from above by $2n + 1$. The Kummer spaces in case of $p = n = 2$ were studied in more detail in [CV13]. The Kummer spaces in case of $p = 3$ and $n = 1$ were classified in [Rac09] and [MV12]. So far the formula $pn + 1$ for the upper bound of the dimension holds in all the known cases, and we conjecture it to be true in general.

In [CGM⁺] the monomial Kummer spaces in division cyclic algebras of prime degrees were classified. Furthermore, an upper bound was provided for the dimension of the Kummer subspaces of the generic cyclic algebra, i.e. the algebra $(\alpha, \beta)_{p, K}$ where K is the purely transcendental field extension of F generated by α and β .

In Section 2 we present a family of maximal Kummer subspaces (with respect to inclusion) of tensor products of cyclic algebras of any degree. These spaces happen also to be monomial. This section is taken from [Cha13] and is based in turn on a result from [Cha09].

In Section 3 we classify the monomial Kummer subspaces of tensor products of cyclic algebras of degree 3. For their description we make use of graph theory. This section is based on results from [Cha13].

In Section 4 we provide an upper bound for the dimension of a Kummer subspace of the generic tensor product of cyclic algebras of degree 3.

2. Maximal Kummer Subspaces of Tensor Products of Cyclic Algebras

$$\text{Fix } A = \bigotimes_{k=1}^n (\alpha_k, \beta_k)_{p, F} = \bigotimes_{k=1}^n F[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_k x_k = \rho x_k y_k].$$

Let $V_0 = F$ and $V_k = F[x_k]y_k + V_{k-1}x_k^{a_k}$ for any $1 < k \leq n$ and a_k prime to p . Assume that $v^p \in F$ for all $v \in V_{k-1}$ for a certain k . Every element of V_k is of the form $f(x_k)y_k + vx_k^{a_k}$ for some $f(x_1) \in F[x_1]$ and $v \in V_{k-1}$. Since v commutes with x_k and y_k , and $y_1x_1 = \rho x_1y_1$, $(f(x_k)y_k + vx_k^{a_k})^p = (f(x_k)y_k)^p + v^p x_k^{pa_k} = N_{F[x_k]/F}(f(x_k))\beta_k + v^p \alpha_k^{a_k} \in F$. For any $1 \leq m \leq p-1$, if $f(x_k) \neq 0$ then the eigenvector of $(f(x_k)y_k + vx_k^{a_k})^m$ corresponding to the eigenvalue ρ^m with respect to conjugation by x is $(f(x_k)y_k)^m$, which is not zero, and therefore $(f(x_k)y_k + vx_k^{a_k})^m \notin F$. If $f(x_k) = 0$ then what is left is $vx_k^{a_k}$, and of course $v^m x_k^{a_k m} \notin F$. Consequently, V_k is Kummer. Since $V_0 = F$, by induction V_k is Kummer for every $1 \leq k \leq n$. The dimension of each V_k is $pk + 1$.

Theorem 2.1. *For any $k \leq n$, V_k is maximal with respect to inclusion.*

Proof. Let $V = V_k$. V has a standard basis

$$B = \{x_i^j y_i x_{i+1} \dots x_k : 1 \leq i \leq k, 0 \leq j \leq p-1\} \cup \{x_1 x_2 \dots x_k\}.$$

Let z be a nonzero element in the algebra A . This element can be expressed as a linear combination of the monomials $x_1^{c_1} y_1^{e_1} \dots x_n^{c_n} y_n^{e_n}$. Let us assume negatively that $V + Fz$ is p -central. Consequently, $w^{p-1} * z \in F$ for every $w \in B$. Since we can subtract from z the appropriate linear combination of the elements of B , we can assume that $w^{p-1} * z = 0$ for every $w \in B$.

Let us pick one monomial $t = x_1^{c_1} y_1^{e_1} \dots x_n^{c_n} y_n^{e_n}$.

If $e_1 = e_2 = \dots = e_n = 0$ then t commutes with $x_1 x_2 \dots x_k \in V$. Since $(x_1 x_2 \dots x_k)^{p-1} * z = 0$, the coefficient of t in z is zero.

Otherwise, let i be the maximal integer for which $e_i \neq 0$. The monomial t commutes with the element $x_i^r y_i x_{i+1} \dots x_k \in V$ where $r \equiv c_i e_i^{-1} \pmod{p}$. Since $(x_i^r y_i x_{i+1} \dots x_k)^{p-1} * z = 0$, the coefficient of t in z is zero.

Since it holds for any monomial t , $z = 0$, and that is a contradiction. \square

3. Monomial Kummer subspaces of Tensor Products of Cyclic Algebras of Degree 3

Keep A as before and assume $p = 3$. Let \mathcal{X} be the set of all Kummer elements in A . We build a directed graph (\mathcal{X}, E) by drawing an edge from y to x

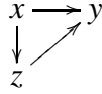
$$y \longrightarrow x$$

if $yxy^{-1} = \rho x$. For any subset $B \subset \mathcal{X}$, (B, E_B) is the subgraph obtained by taking the vertices in B and all the edges between them. The set B is called ρ -commuting

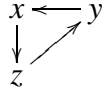
if it is linearly independent over F and for every two distinct elements $x, y \in B$, $xyx^{-1} = \rho^k k$ for $k \in \{0, 1, 2\}$. In particular, any set of monomials is ρ -commuting.

According to [CV12, Corollary 2.2], a set $\{x_1, \dots, x_m\}$ spans a Kummer space if and only if every subset of cardinality three $\{x_i, x_j, x_k\}$ spans a Kummer space. Therefore we will start with the set of cardinality 3.

Lemma 3.1. *Given a ρ -commuting set $\{x, y, z\}$, $Fx + Fy + Fz$ is Kummer if and only if either*



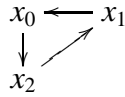
or $xyz \in F$, in which case



Proof. If $xy = yx$ then $x^2 * y = 3x^2 y \in F$ which means that $y \in Fx$, contradiction. Consequently we are left with the two graphs above (up to a change in the order of the elements). In the first case, $x * y * z = 0$, so there are no extra conditions. In the second case, $x * y * z = -3\rho^{-1}xyz \in F$. The opposite direction is a straight-forward computation. \square

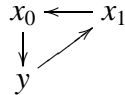
Let B be a ρ -commuting set spanning a Kummer space. We will now study the properties of the directed graph (B, E_B) . By a cycle we always mean a **simple directed cycle**.

Proposition 3.2. *If (B, E_B) contains a cycle of length 3*



then for every $y \in B \setminus \{x_0, x_1, x_2\}$, either $x_k \longrightarrow y$ for any $k \in \{0, 1, 2\}$ or $x_k \longleftarrow y$ for any $k \in \{0, 1, 2\}$.

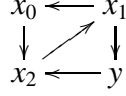
Proof. If $x_0 \longrightarrow y$ and $x_1 \longleftarrow y$ then



which means that $yx_0x_1 \in F$. Since $x_0x_1x_2 \in F$, we get $y \in Fx_2$, which contradicts the linear independence. The rest of the proof repeats the same idea. \square

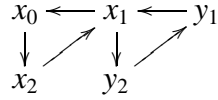
Proposition 3.3. *The cycles of (B, E_B) are vertex-disjoint.*

Proof. First assume that

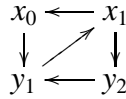


Then $yx_1x_2 \in F$ whereas $x_0x_1x_2 \in F$, which means that $y \in Fx_0$, and that contradicts the linear independence.

Assume that



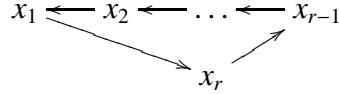
From Proposition 3.2 we have $x_0 \longrightarrow y_2$ and $y_1 \longrightarrow x_0$. But then



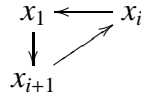
and we saw already that this is impossible. \square

Proposition 3.4. *There are no cycles of length greater than 3.*

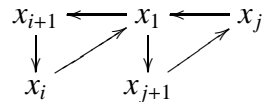
Proof. Assume



for some $r \geq 4$. Let i be the maximal integer between 1 and r such that $x_i \longrightarrow x_1$. Now, $x_1 \longrightarrow x_{i+1}$. Therefore



If $i \geq 3$ then according to Proposition 3.2, $x_1 \longrightarrow x_{i-1}$, which implies that $i \neq 3$, or in other words $i \geq 4$. Let j be the minimal index for which $x_1 \longrightarrow x_{j+1}$. In particular $x_j \longrightarrow x_1$. Now, $j+1 \leq i-1$, which means that



But this is impossible. If $i = 2$ then according to Proposition 3.2, $x_4 \longrightarrow x_1$ which contradicts the maximality of i . \square

As a consequence we obtain the following theorem:

Theorem 3.5. *A ρ -commuting subset B of X spans a Kummer space if and only if the graph (B, E_B) satisfies the following axioms:*

1. *For every two distinct elements $x, y \in B$, either $x \longrightarrow y$ or $x \longleftarrow y$*
2. *All cycles are of length 3.*
3. *The product of all the elements in a cycle is in F .*
4. *The cycles are vertex-disjoint.*

Proof. The straight-forward direction is an immediate result of what we did so far. The opposite direction is a result of the fact that every three elements in this set span a Kummer space according to Lemma 3.1. \square

Corollary 3.6. *Given a ρ -commuting set B spanning a Kummer space, if $\#B = m$ then the longest path $x_1 \longrightarrow x_2 \longrightarrow \dots \longrightarrow x_r$ in the graph (B, E_B) satisfying $x_i \longrightarrow x_j$ for any $1 \leq i < j \leq r$ is of length no less than $m - \lfloor \frac{m}{3} \rfloor$.*

Proof. Take B and take off exactly one element from each cycle. The number of elements taken off is at most $\lfloor \frac{m}{3} \rfloor$, and what is left satisfies the required condition. \square

Corollary 3.7. *The maximal ρ -commuting set spanning a Kummer space in A is of cardinality $3n + 1$.*

Proof. We are already familiar with monomial Kummer spaces of size $3n + 1$. According to the previous corollary, if we have a ρ -commuting set B of size $3n + 2$ spanning a Kummer space then we have a path in (B, E_B)

$$x_1 \longrightarrow x_2 \longrightarrow \dots \longrightarrow x_{2n+2}$$

satisfying $x_i \longrightarrow x_j$ for any $1 \leq i < j \leq 2n + 2$. Then the set B generates over F a tensor product of $n + 1$ cyclic algebras of degree 3

$$F[x_1, x_2] \otimes F[x_1 x_2^{-1} x_3, x_1 x_2^{-1} x_4] \otimes \dots \otimes F[(\prod_{k=1}^n x_{2k-1} x_{2k}^{-1}) x_{2n+1}, (\prod_{k=1}^n x_{2k-1} x_{2k}^{-1}) x_{2n+2}],$$

contradiction. \square

4. The Generic Tensor Product of Cyclic Algebras

Let K be a purely transcendental field extension of F generated by $\{\alpha_k, \beta_k : 1 \leq k \leq n\}$. Fix $A = \bigotimes_{k=1}^n K[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_k x_k = \rho x_k y_k]$.

Theorem 4.1. *For any Kummer subspace of A there exists a monomial Kummer space of the same dimension.*

Proof. Write $V = Fv_1 + Fv_2 + \dots + Fv_m$, where n is the dimension of V . Each v_k is a sum of monomials of the form $cx_1^{a_1}y_1^{b_1} \dots x_n^{a_n}y_n^{b_n}$ where the coefficient c is a quotient of two polynomials in the variables $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ over F . By multiplying by all the denominators, we can assume that the coefficients are polynomials. Then each v_k can be written also as a polynomial in the variables $x_1, y_1, \dots, x_n, y_n$ over F . We now impose a lexicographical valuation on the polynomials in $F[x_1, y_1, \dots, x_n, y_n]$. In particular, every polynomial has now a leading monomial, i.e. the monomial $x_1^{a_1}y_1^{b_1} \dots x_n^{a_n}y_n^{b_n}$ of the highest value with a nonzero coefficient.

By the following process we can make sure the leading monomials of v_1, \dots, v_m are distinct and linearly independent over K : If the leading monomial of v_1 is $x_1^{a_1}y_1^{b_1} \dots x_n^{a_n}y_n^{b_n}$ then we take the coefficient c of $x_1^{a'_1}y_1^{b'_1} \dots x_n^{a'_n}y_n^{b'_n}$ in v_1 when writing v_1 as a polynomial in $(F[\alpha_1, \beta_1, \dots, \alpha_n, \beta_n])[x_1, y_1, \dots, x_n, y_n]$ where a'_k, b'_k are the unique integers between 0 and $p-1$ satisfying $a'_k \equiv a_k, b'_k \equiv b_k \pmod{p}$. Then for each $2 \leq i \leq m$ we replace each v_i with $cv_i - c_i v_1$ where c_i is the coefficient of that monomial in v_i . Then we fix v_2 and change v_3, \dots, v_m similarly, and so on.

Let w_k be the leading monomial of v_k . For any set of nonnegative integers d_1, \dots, d_m satisfying $d_1 + \dots + d_m = p$, The expression $w_1^{d_1} * \dots * w_m^{d_m}$ is either equal to the leading monomial of $v_1^{d_1} * \dots * v_m^{d_m}$ or to zero. In both cases, it is in F , which means that $Fw_1 + \dots + Fw_m$ is monomial Kummer. \square

Corollary 4.2. *If $p = 3$ then the upper bound for the dimension of a Kummer subspace of A is $3n + 1$.*

Proof. Follows immediately from the previous theorem and Section 3. \square

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